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BAYES MARKOVIAN DECISION MODELS
FOR A MULTISTAGE REJECT ALLOWANCE PROBLEM

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REJECT ALLOWANCE PROBLEM

by

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1. Introduction

Consider the situation where a production system is required to have a given number of non-defective items available for shipment at a specified future time. When the production time span can be broken up into a sequence of periods and when the production process is assumed to be perfect (no defectives are produced), then the production planning problem can be viewed in terms of allocating a scarce resource (produced items) to the several periods [9]. The optimal production schedule will depend on the production cost function and inventory storage cost in each period, and possibly on the costs of changing production rates from period to period.

A production process, however, is most often imperfect and sometimes the production time span is not naturally divided into periods. When the first or both of these conditions hold, the production planning problem becomes one of scheduling production in excess of the required amount so

that a sufficient number of non-defective items will be available for shipment. The excess production is termed the reject (scrap or shrinkage) allowance.

The reject allowance problem was first formulated as a static problem in which an optimal production rate is determined prior to the period of production and the output is inspected after production is completed. Bowman [1], in an early study, approached the problem under the assumption that process quality is independent of the number of units produced. Goode and Saltzman [6] and Hillier [7] have also made this assumption in connection with a different version of the static problem. On the other hand, Giffler [5] and Levitan [10] have approached the static problem under the assumption that process quality is a function of the production rate.

Klein [8], in a recent paper, seems to have been the first to formulate dynamic (multistage) versions of the reject allowance problem. He assumed that inspection can be performed at intermediate points within the production time span as well as at the conclusion of production. In one model the time span is divided into a fixed number of subspans (stages) not necessarily of equal length. This would be the case where the time span could be divided into a sequence of periods when (say) the needed equipment was available. In a second model, the number and length of the stages are random variables. In the first model, the stages are assumed to be independent in the sense that process quality in any stage is taken to be a function only of that stage and the amount produced during the stage. In the second model, the probability of producing a defective in

in any stage is assumed to be a function of the number of items previously produced, the cumulative number of non-defectives and the number of items produced during that stage. In this model, the costs of production, inspection, storage, and the excess and shortage charges are also functions of this data. In the first model, these costs are related to the stage, the cumulative number of non-defectives, and the production rate for the stage. Thus in both models, the functions describing process quality and costs are quite general. Using the theory of Markovian sequential decision processes as developed by Derman [2], [3] and Derman and Klein [4], Klein shows that optimal decision rules for scheduling production for both models may be determined by linear programming methods. In addition, he shows how the linear programming formulation allows for the inclusion of various side conditions.

The general multistage models for the reject allowance problem that Klein has formulated are left open to specific interpretation. In this paper we define a particular reject allowance problem and analyze several versions of it using Klein's models and some fundamental notions of statistical decision theory [11], [12], as starting points. The problem we shall consider arises when constraints on equipment and manpower (hereafter referred to as production capacity) are specified prior to the start of production. In the first model, production capacity is specified (or determined) at the beginning of production and remains fixed for the entire run. The model can be described as a Bayesian multistage analog to Bowman's problem. In the second model, there is an upper bound on production

capacity, but after the start of production the actual production rate may be increased or decreased at specified future times (possibly at a cost). This model can be viewed as a Bayesian analog to Klein's first model.

2. A Bayes Markovian Decision Model for Fixed Capacity

Suppose that an order for I items is required for shipment at a known future date. At the start of production a certain amount of production capacity K , $K \geq I$, is available. The quality of produced items is assumed to be stable in the sense that the production process is viewed as a Bernoulli process [11] with parameter p , the probability of producing a defective. It will not, however, be assumed that p is known. Instead, p will be considered a random variable¹ (in the Bayesian sense) with a prior density function $f_{\beta}(\cdot | i_0, r_0)$ from the beta family, where

$$(2.1) \quad f_{\beta}(p | i_0, r_0) = \frac{(r_0-1)!}{(i_0-1)!(r_0-i_0-1)!} (1-p)^{i_0-1} p^{r_0-i_0-1},$$

$$i_0, r_0 = 0, 1, 2, \dots, \quad 0 \leq p \leq 1.$$

¹Following Raiffa and Schlaifer [12] we denote random variables with tildes, e.g., the random variable corresponding to the parameter p is written $\tilde{p}$.

Under the fixed capacity assumption, Klein's second model which provides for an arbitrary number of inspection points (stages) seems most appropriate. This model allows information to be accumulated during the production run. The information is used to determine the cumulative number of non-defectives produced thus far and to revise judgment about the distribution of $\tilde{p}$ via Bayes' theorem. The cumulative number of non-defectives and the posterior distribution of $\tilde{p}$ are then used in our adaptation of the model to find the optimal number of items to produce before the next inspection. Actually, the linear programming formulation for this problem allows the optimal sequence of production levels to be determined simultaneously.

Klein's second model, particularized to allow for our Bayesian interpretation, is formulated as follows.¹ At any time during the production run, the production results thus far can be described by the number of items produced r , $r = 0, 1, \dots, K$, and the number of non-defectives, i_r , among the r items. Thus, the observed evolution of the production process can be described in terms of the state space S where

$$S = \{i_r : i_r \leq r, r=0, 1, \dots, K\},$$

K being the fixed production capacity. The process always begins in state 0_0 . At the end of the first stage, the process will be in state j_x if the

¹Wherever possible we retain the notation of Klein [8].

first production decision resulted in the production of x items and j_x of these were found to be non-defective. The process will continue to evolve in this way until state i_K is reached or until it is decided to stop production.

As in [8], each decision by which x , the number of units to produce, is chosen may be a randomized decision over the possible values of x . Letting $d(i_r, x)$ represent the conditional probability of selecting production level x given that the process is in state i_r , we denote any randomized decision rule by a matrix $\underline{D}$ with elements

$$d(i_r, x) \geq 0,$$

and such that

$$\sum_{x=0}^{K-r} d(i_r, x) = 1.$$

Now let $\{\tilde{y}_n: n=1, 2, \dots, \tilde{N}+1\}$ denote the sequence of states observed at the start of stages $n=1, 2, \dots, \tilde{N}$; $y_{\tilde{N}+1}$ denotes the final state of the process. (Note that $\tilde{N}$ is a random variable through its dependence on $\underline{D}$.) Also, let $\{\tilde{\Delta}_n: n=1, 2, \dots, \tilde{N}\}$ denote the sequence of production decisions. Then we define

$$(2.2) \quad q_{ij}(r, x) \equiv P\{\tilde{y}_{n+1}=j_{r+x} \mid \tilde{y}_n=i_r, \tilde{\Delta}_n=x\}, \quad n=1, 2, \dots, \tilde{N}.$$

Thus, $q_{ij}(r, x)$ denotes the probability that $(j-i)$ additional non-defectives will result from the production of x additional items, $x \leq K-r$. Therefore, when $P\{\tilde{y}_1=0_0\} = 1$ and any decision rule $\underline{D}$ is used, the sequence of observed states $\{\tilde{y}_n\}$ is a Markov chain with stationary transition probabilities

$$p(i_r, j_s) = d(i_r, x) q_{ij}(r, x), \quad s = r+x, i_r \in S-T,$$

where

$$T = \{i_r : i_r \geq I \text{ or } r = K\}$$

represents the set of "stopping" states. We make the Markov chain cyclical by setting

$$q_{i0}(r, 0) = 1, \quad i_r \in T$$

so that

$$p(i_r, 0_0) = 1, \quad i_r \in T.$$

Also, to take care of the situation where the decision to terminate production precedes entry into a stopping state, we let

$$q_{i0}(r, 0) = 1, \quad i_r \in S-T$$

and thus

$$p(i_r, 0_0) = d(i_r, 0), \quad i_r \in S-T.$$

At this point we introduce a Bayesian approach to the evaluation of $q_{ij}(r, x)$, $i_r \in S-T$, $j_{r+x} \in S$, $x > 0$. First note that if p , the probability of producing a defective is assumed to be known, then $(j-i)$ would have a binomial distribution with parameters x and p , i.e.,

$$q_{ij}(r, x) \equiv f_b(j-i | p, x) \equiv \binom{x}{j-i} (1-p)^{j-i} p^{x-j+i}.$$

However, we have assumed p to be a random variable with a beta prior given by equation (2.1). The "sampling process" (producing x items and finding $j-i$ non-defectives) is binomial and, consequently, $q_0(0,x)$ is a beta-binomial mass function for $\tilde{j}$ (see Raiffa and Schlaifer [12], page 265) where,

$$q_{0j}(0,x) \equiv \int_0^1 f_b(j|p,x) f_\beta(p|i_0,r_0) dp, \quad x = 1,2,\dots,K; j \leq x.$$

Moreover, using Bayes' theorem, it can be shown that in general,

$$(2.3) \quad q_{ij}(r,x) = \int_0^1 f_b(j-i|p,x) f_\beta(p|i_0+i,r_0+r) dp, \quad i_r \in S-T, j_{r+x} \in S; x > 0,$$

$$= \frac{(j+i_0-1)!(r_0+r+x-j-i_0-1)! x!(r_0+r-1)!}{(j-i)!(i_0+i-1)!(x-j-i)!(r_0+r-i_0-i-1)!(r_0+r+x-1)!}$$

The second expression appears in Raiffa and Schlaifer [12], page 237.

Now for fixed capacity K , let $c(i_r,x)$ denote the total cost of producing and inspecting x items and storing i_r items during this stage of production. If $i_r \in T$ or $x = 0$, then $c(i_r,x)$ also includes the cost of excess if $i_r > I$ or shortage if $i_r < I$, and the remaining storage costs until the date of shipment.

The description of the fixed capacity problem is now complete. Using the results of Derman [2], we can formulate the problem as a linear program.

The solution to the linear program is then used to find the optimal decision rule. (As pointed out in [4], [8], the problem could also be formulated as a dynamic program.)

The linear program to be solved is given by:

$$\text{Minimize: } Z = \sum_i \sum_x z(i_r, x) c(i_r, x)$$

$$\text{Subject to: } z(j_s, x) \geq 0, \quad j_s \in S, x = 0, 1, \dots, k-s;$$

$$z(*) \geq 0 \quad ;$$

$$\sum_{j_s} \sum_x z(j_s, x) - z(*) = 0 \quad ;$$

$$\sum_x z(j_s, x) - \sum_{i_r} z(i_r, s-r) q_{ij}(r, s-r) = 0, \quad j \geq i, s \geq r, j_s \in S-0_0;$$

$$\sum_x z(0_0, x) - \sum_{i_r \in T} \sum_x z(i_r, x) = 0 \quad ;$$

$$\sum_x z(0_0, x) = 1.$$

The elements of the optimal decision rule are computed as

$$d(i_r, x) = \frac{z(i_r, x)}{\sum_x z(i_r, x)}$$

If the denominator is zero then the value of d may be set arbitrarily without changing the expected minimum cost.

It might be assumed that prior to the start of production several different production capacities are available rather than just one. In this case, let K_0 denote the set of capacities and $C(K)$, $K \in K_0$, the costs related to operating at capacity K . Then the optimal capacity K^* could be determined by solving the equation

$$C(K^*) = \min_{K \in K_0} \{c(K) + Z(K)\}$$

where $Z(K)$ is the linear programming solution for capacity K .

3. A Bayes Markovian Decision Model for Variable Capacity

In this section we assume a maximum production rate k . We also assume that the actual production rate x'_n , $x'_n \leq k$, may be increased or decreased at $N-1$ fixed points during the production time span. Let t_n , $n = 1, 2, \dots, N$ denote the length of the n^{th} stage. Then the maximum number of units that can be produced in stage n is given by K_n where

$$K_n = kt_n.$$

The actual number produced will be given by x_n , $x_n \leq K_n$. Thus the production rate, x'_n , for stage n is given by

$$x'_n = x_n/t_n.$$

Finally, we shall assume that the cost of changing production rates from stage n to $n+1$ is given by the known function

$$h(x'_n, x'_{n+1}), \quad n = 1, \dots, N; \quad x'_n, x'_{n+1} \leq k.$$

As an example of where the variable capacity model might apply consider the case where production equipment is rented for fixed periods of time (e.g., by the week or month), and where decisions on changing capacities are made at the end of each rental period. The assumption of an upper limit on capacity could result from a limit on available skilled labor or from a limit on available floor space.

In order to describe the evolution of the production process for the variable capacity problem, we specify the state space

$$S = \{(r_{n-1}, x_{n-1}, i_n) : x_{n-1} \leq K_{n-1}; r_{n-1} = \sum_{j=1}^{n-2} x_j; i_n \leq r_{n-1} + x_{n-1}; n = 1, 2, \dots, N+1\},$$

where i_{N+1} is the cumulative number of non-defectives produced during N stages and where $r_0 = r_1 = x_0 = i_1 = 0$. (Note that r_n denotes the cumulative number of items produced during the first $n-1$ stages, i.e., $r_n = r_{n-1} + x_{n-1}$.) To make the state notation slightly less cumbersome let $i_{r,x}^{(n)} = (r_{n-1}, x_{n-1}, i_n)$; thus, for example, the starting state $(0,0,0)$ is denoted $0_{0,0}(1)$.

Again we allow for randomized decision rules $\underline{D}$ with elements now written as $d(i_{r,x}(n), x_n)$, $n = 1, \dots, N$. Also as before we let $\{\tilde{y}_n: n=1, 2, \dots, N+1\}$ denote the sequence of observed states and $\{\tilde{\Delta}_n: n=1, 2, \dots, N\}$, the sequence of production decisions. Then similar to equation (2.2) we define

$$(3.1) \quad q_{ij}(r_n, x_n) \equiv P\{\tilde{y}_{n+1}=j_{r,x}(n+1) | \tilde{y}_n=i_{r,x}(n), r_n = r_{n-1} + x_{n-1}, \tilde{\Delta}_n = x_n\}, \quad n = 1, 2, \dots, N+1.$$

Thus, when $P(\tilde{y}_1 = 0_{0,0}(1)) = 1$ and any rule $\underline{D}$ is used, the sequence $\{\tilde{y}_n\}$ is a Markov chain with stationary transition probabilities,

$$p[i_{r,x}(n), j_{r,x}(n+1)] = d(i_{r,x}(n), x_n) q_{ij}(r_n, x_n), \quad n = 1, 2, \dots, N.$$

To return the process to its starting state we set

$$p[i_{r,x}(n+1), 0_{0,0}(1)] = 1, \quad \text{all } i_{r,x}(n+1).$$

Then $\{\tilde{y}_m: m=1, 2, \dots\}$; where for $m > N+1$, $m=n \bmod(N+1)$, $n=1, 2, \dots, N+1\}$ is a cyclical Markov chain with one ergodic class of states.

To give the variable capacity model a Bayesian interpretation we again characterized the production process as a Bernoulli process with unknown parameter p . Then if $\tilde{p}$ is assumed to have a beta prior, equation (2.1), with parameters i and r , $q_{ij}(r_n, x_n)$ is evaluated as,

$$(3.2) \quad q_{ij}(r_n, x_n) = \int_0^1 f_b(j_{n+1}-i_n | p, x_n) f_\beta(p | i+i_n, r+r_n) dp, \quad n = 1, 2, \dots, N.$$

The solution of the integral in equation (3.2) has been given in equation (2.3).

The cost associated with the n^{th} stage of the process is denoted by $c(i_{r,x}(n), x_n)$, $n = 1, 2, \dots, N+1$. This cost for $n \leq N$ will be the sum of the costs of producing and inspecting x_n items, the cost of storing i_n items for time t_n , and the cost of changing the production rate, $h(x'_{n-1}, x_n)$. For $n = N+1$, $c(i_{r,x}(N+1), x_{N+1})$ denotes the excess or shortage cost associated with the total production of i_{N+1} non-defective items.

The Bayes' variable capacity model described above includes aspects of both of Klein's models, [8]. Its formulation as a linear program is similar to that of Klein's first model and follows from Derman and Klein [4]. The linear program is written as:

$$\begin{aligned}
 &\text{Minimize:} && Z = (N+1) \sum_n \sum_{i_{r,x}} (n) \sum_{x_n} z(i_{r,x}(n), x_n) c(i_{r,x}(n), x_n) \\
 &\text{Subject to:} && z(i_{r,x}(n), x_n) \geq 0, \quad \text{all } i_{r,x}(n) \in S, \quad x_n \leq K_n; \\
 (3.3) \quad &&& \sum_{x_{n+1}} z(j_{r,x}(n+1), x_{n+1}) - \sum_{i_{r,x}} (n) z(i_{r,x}(n), x_n) q_{ij}(r_n, x_n) = 0, \\
 &&& j_{r,x}(n+1) \in S, \quad n = 1, 2, \dots, N-1; \\
 &&& \sum_{x_1} z(0_{0,0}(1), x_1) - \sum_{i_{r,x}} (N+1) \sum_{x_{N+1}} z(i_{r,x}(N+1), x_{N+1}) = 0; \\
 &&& \sum_n \sum_{i_{r,x}} (n) \sum_{x_n} z(i_{r,x}(n), x_n) = 1.
 \end{aligned}$$

As before, the elements of the optimal rule $\underline{D}$ are given by

$$d(i_{r,x}^{(n)}, x_n) = \frac{z(i_{r,x}^{(n)}, x_n)}{\sum_{x_n} z(i_{r,x}^{(n)}, x_n)}.$$

4. Discussion

Klein [8] has shown that the linear programming formulation of a multistage reject allowance problem allows for the possibility of including certain side conditions. Thus, in our second model, for example, if it were desired that the expected number of non-defectives produced in the first $m-1$ periods, $m = 2, \dots, N+1$, be greater than I_m , we could include this condition by adding the constraint

$$(4.1) \quad \sum_{r,x}^{(N+1)} z(i_{r,x}^{(m)}, x_m) i_m > I_m$$

to the linear program (3.3). Derman [2] has shown that the linear programming solution to a Markovian decision model with no side conditions always leads to a non-randomized optimal decision rule. But when side conditions are added, the optimal rule may be randomized.

However, whether or not the optimal rule is randomized, it may be very difficult to calculate. This is due to the fact that the linear program associated with a Markovian decision model is often unreasonably large. In some cases, it may be possible to use decomposition methods [14]

to overcome this "curse of dimensionality." In other cases, the nature of transitions from state to state plus the form of the objective function may allow for a reduction of the state space, [13].

It is clear in the present case that both of our models can lead to computational difficulties. As a way to reduce the number of states in the first model, one could limit production decisions, x , to be multiples of a basic production level (number of units) x_0 . Thus, for example, if $x_0 = 10$ units, x is chosen from the set $\{10, 20, 30\}$ and $K = 30$, then the state space

$$S = \{0_0, 10_0, \dots, 10_{10}, 20_0, \dots, 20_{20}, 30_0, \dots, 30_{30}\}$$

would obtain. This state space would include 64 states rather than 496.¹

The idea of limiting decisions to multiples of a basic production level may be valid when it is only feasible (or practical) to conduct inspections at equally spaced time intervals, e.g., at the end of each week. The basic production level would then correspond to a week's production.

¹In general the reduction would be from $\frac{(K+1)(K+2)}{2}$ states to

$1 + \frac{K}{2x_0} [K+x_0+2]$ when $\frac{K}{x_0}$ is an integer.

In the second model it might be possible to justify the use of a basic production rate x'_0 in determining the set of production decisions to be considered. The production decision in each period would then be limited to integer multiples of this rate within capacity restrictions.

The Bayesian approach seems very natural in relation to our models. In fact, it should be possible to give a Bayesian interpretation to any Markovian decision model in which the states represent accumulated information from the system. As another example, consider the following sketch of a simple "stopping rule" problem. Suppose a machine is set up to produce K items; after the K^{th} item is produced the machine must be broken down and set up again. After each set up the machine will produce defectives at a rate p_1 if set up correctly and a rate $p_2 > p_1$ otherwise. Let the states of the Markovian decision model be given by $\{i_r: i_r < r; r = 0, \dots, K\}$ where r is the number of produced items and i_r the number of non-defectives since the last set up. To keep things simple, suppose that after every x_0 items are produced it must be decided either to reset the machine or to produce x_0 additional items under the present set up.

The problem may be formulated in terms of minimizing the average cost of setup, production and repair per unit produced and the Bayesian analysis would enter into the evaluation of the probability of going from state i_r to j_{r+x_0} given the decision is to produce x_0 items. If we let $f'(p_1)$ and $f'(p_2)$ be the prior probabilities associated with p_1 and p_2 and define

$q_{ij}(r, x_0)$ as in equation (2.2), we would have

$$\begin{aligned} q_{ij}(r, x_0) &= q_{ij}(r, x_0 | p_1) P(p_1 | i_r) + q_{ij}(r, x_0 | p_2) P(p_2 | i_r) \\ &= C[(1-p_1)^j p_1^{r+x_0-j} f'(p_1) + (1-p_2)^j p_2^{r+x_0-j} f'(p_2)] \end{aligned}$$

where C is a normalizing constant independent of p_1 and p_2 . The values of $q_{ij}(r, \cdot)$ corresponding to the decision to reset the machine would be

$$q_{i0}(r, 0) = 1, \quad i \geq 0, r \leq K$$

$$q_{ij}(r, 0) = 0, \quad i, j \geq 0, r \leq K.$$

The linear programming formulation would be similar to the one used with our first model.

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